Energy Localization in Granular Crystals for Energy Harvesting

CAL POLY SAN LUIS OBISPO

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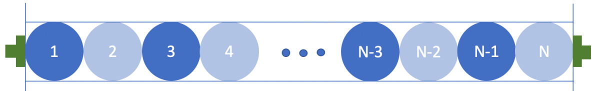

Figure 1: Granular chain

Set Up

The focus of this project is the study of time-periodic solutions on a dimer granular crystal chain. We consider a chain composed of $N = 20$ spherical beads which are made of chrome steel (dark blue particles in Figure 1) and tungsten carbide (light blue particles in Figure 1) materials (see material parameters in Table 1). The beads are supported by two rods which restrict their lateral movements. A static force, denoted as F_0 hereafter, is applied at both ends in order to make sure each bead is in contact with adjacent beads. The granular chain is harmonically driven at the boundaries corresponding to 0^{th} and $(N+1)^{th}$ location of the chain. The drive at both ends has the form $a\cos(2\pi f t)$ where a and f are the amplitude and frequency of the excitation, respectively.

Material	r (mm)	$E\ (Pa)$	v	$\rho\ (\frac{kg}{m^3})$	$M\ (g)$
Chrome Steel	9.525	$200 * 10^9$	0.27	7780	28.16
Tungsten Carbide	9.525	$668 * 10^9$	0.24	15800	57.19

Table 1: Material parameters

Equations of Motion

The granular crystal chain is modeled by

$$\ddot{u}_n = \frac{A_{n-1}}{Mn}\left[\delta_{0,n-1} + u_{n-1} - u_n\right]_+^{\frac{3}{2}} - \frac{A_n}{M_n}\left[\delta_{0,n} + u_n - u_{n+1}\right]_+^{\frac{3}{2}} - \frac{\dot{u}_n}{\tau}$$

with $n = 1,2,\ldots,N$. The parameters and variables are presented in Table 2. We break down this second-order ordinary differential equation into a system of two first-order differential equations

$$\begin{cases} \dot{u}_n = p_n \\ \dot{p}_n = \frac{A_{n-1}}{M_n}\left[\delta_{0,n-1} + u_{n-1} - u_n\right]_+^{\frac{3}{2}} - \frac{A_n}{M_n}\left[\delta_{0,n} + u_n - u_{n+1}\right]_+^{\frac{3}{2}} - \frac{p_n}{\tau} \end{cases}$$

u_n	Displacement of the n^{th} bead	A_n	Elastic coefficient of the n^{th} bead
$\dot{u}_n$	Velocity of the n^{th} bead	M_n	Mass of the n^{th} bead
$\ddot{u}_n$	Acceleration of the n^{th} bead	$\delta_{0,n}$	Precompression factor of the n^{th} bead ($F_0 = A_n\delta_{0,n}^{3/2}$)
$[x]_+$	$[x]_+ = \max(0,x)$	τ	Dissipation parameter

Table 2: Variables

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This final system of $2N$ equations can be written in the form $\dot{\vec{x}} = \vec{F}(t,\vec{x})$ with $\vec{x} = [u_1, u_2, \ldots, u_N, p_1, p_2, \ldots, p_N]^T$

and

$$\vec{F}(t,\vec{x}) = \begin{bmatrix} p_1 \\ \vdots \\ p_N \\ \frac{A_0}{M_1}\left[\delta_{0,0} + u_0 - u_1\right]_+^{\frac{3}{2}} - \frac{A_1}{M_1}\left[\delta_{0,1} + u_1 - u_2\right]_+^{\frac{3}{2}} - \frac{p_1}{\tau} \\ \vdots \\ \frac{A_{N-1}}{M_N}\left[\delta_{0,N-1} + u_{N-1} - u_N\right]_+^{\frac{3}{2}} - \frac{A_N}{M_N}\left[\delta_{0,N} + u_N - u_{N+1}\right]_+^{\frac{3}{2}} - \frac{p_N}{\tau} \end{bmatrix}.$$

Linear Problem and Normal Modes

Assume a very small amplitude a is used which leads to a relatively small displacement, so

$$\frac{|u_n - u_{n+1}|}{\delta_{0,n}} \ll 1$$

Using a Taylor expansion, the equations of motion become

$$\ddot{u}_n = \frac{K_{n-1}}{M_n}(u_{n-1} - u_n) - \frac{K_n}{M_n}(u_n - u_{n+1}) - \frac{\dot{u}_n}{\tau}$$

with $K_n = \frac{3}{2}A_n\delta_{0,n}^{\frac{1}{2}}$.

Thus we can write the system of $2N$ first-order differential equations in matrix form

$$\dot{Y} = \mathcal{A}Y$$

with $Y = [u_1, \ldots, u_N, p_1, \ldots, p_N]^T$ and

$$\mathcal{A} = \begin{pmatrix} \mathbb{O} & \mathbb{I} \\ \mathcal{B} & \left(-\frac{1}{\tau}\right)\mathbb{I} \end{pmatrix}$$

where $\mathbb{O}$ and $\mathbb{I}$ is the $N \times N$ zero and identity matrices respectively, and

$$\mathcal{B} = \begin{pmatrix} -\frac{K_0+K_1}{M_1} & \frac{K_1}{M_1} & 0 & \cdots & 0 \\ \frac{K_1}{M_2} & -\frac{K_1+K_2}{M_2} & \frac{K_2}{M_2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \frac{K_{N-2}}{M_{N-1}} & -\frac{K_{N-2}+K_{N-1}}{M_{N-1}} & \frac{K_{N-1}}{M_{N-1}} \\ 0 & \cdots & 0 & \frac{K_{N-1}}{M_N} & -\frac{K_{N-1}+K_N}{M_N} \end{pmatrix}$$

We solve the above system of ODEs by using $Y_n = y_n e^{i\omega t}$. This results in an eigenvalue problem of the form of $\mathcal{A}y = \lambda y$. Figure 2 presents the eigenvalues (normal modes) where $f = 2\pi\omega$.

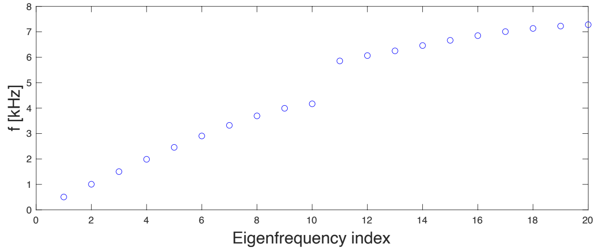

Figure 2: Eigenvalues from the linear problem

4th-Order Runge-Kutta (RK4)

RK4 is a numerical iterative method for approximating solutions to initial value problems (IVPs). For a general IVP $\dot{\vec{x}} = \vec{F}(t,\vec{x})$ with $\vec{x} = \vec{x}(t), \vec{x}(t_0) = \vec{x}_0$, and $t_n = t_0 + n\,h$, where h is the time step, we have the following:

$$\begin{aligned} \vec{k}_1 &= h\,\vec{F}(t_n, \vec{x}_n) \\ \vec{k}_2 &= h\,\vec{F}\left(t_n + \frac{h}{2}, \vec{x}_n + \frac{\vec{k}_1}{2}\right) \\ \vec{k}_3 &= h\vec{F}\left(t_n + \frac{h}{2}, \vec{x}_n + \frac{\vec{k}_2}{2}\right) \\ \vec{k}_4 &= h\vec{F}(t_n + h, \vec{x}_n + \vec{k}_3) \\ \vec{x}_{n+1} &= \vec{x}_n + \frac{1}{6}(\vec{k}_1 + 2\vec{k}_2 + 2\vec{k}_3 + \vec{k}_4) \end{aligned}$$

Fixed Point Method for Time-Periodic Solutions: Newton's Method Approach

Set up the Poincaré map

$$P\left(\vec{x}^{(0)}\right) = \vec{x}^{(0)} - \vec{x}(T)$$

where $\vec{x}(T)$ is the result of integrating $\dot{\vec{x}} = \vec{F}(t,\vec{x})$ from t = 0 to $t = T$ by using RK4 method. Apply Newton's method to the map P to get

$$\vec{x}^{(0,k+1)} = \vec{x}^{(0,k)} - [J]^{-1}_{\vec{x}^{(0,k)}} P\left(\vec{x}^{(0,k)}\right), \qquad k = 0,1,\ldots$$

where $\vec{x}^{(0,k)}$ is the k^{th} iterate, and J is the Jacobian matrix of P given by

$$J = \frac{\partial}{\partial \vec{x}^{(0)}}\left[\vec{x}^{(0)} - \vec{x}(T)\right] = \mathbb{I} - \frac{\partial \vec{x}(T)}{\partial \vec{x}^{(0)}}.$$

Since the Jacobian is not available, we then define

$$V(t) = \frac{\partial \vec{x}(t)}{\partial \vec{x}^{(0)}}$$

and formally differentiate $\dot{\vec{x}} = \vec{F}(t,\vec{x})$ with respect to $\vec{x}^{(0)}$. This way, we obtain the variational problem

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\partial \vec{F}}{\partial \vec{x}}V$$

with initial conditions $V(0) = \mathbb{I}$ and Jacobian of $\vec{F}$ given by $\frac{\partial \vec{F}}{\partial \vec{x}} = D_{\vec{x}}\vec{F}$. V is the principal fundamental matrix solution, and $V(t = T) = M$ corresponds to the monodromy matrix. Its eigenvalues denoted by λ are called the Floquet multipliers and determine the stability trait of the time-periodic solutions identified via Newton's method. In particular, if $|\lambda| > 1$, this will signal a dynamical instability of the pertinent waveform. Otherwise, i.e., $|\lambda| \le 1$, the solution is deemed to be stable. Its eigenvalues are called the Floquet multipliers, which determine the stability of time periodic solutions.

Numerical Result: Frequency continuation

To achieve our goal of matching the maximum velocity with the resonant frequencies, we performed a frequency continuation for $a = 10^{-10}$ where we stored the maximum velocity of a representative bead for each frequency as this shown in Figure 3 with a solid black line. Referring to the linearized problem above, we then found the eigenvalues of matrix $\mathcal{A}$ (shown as vertical solid red lines in Figure 3). We successfully confirmed that the frequencies at which the maximum velocities occurred matched these eigenvalues. Representative examples of time-periodic solutions for two cases: close to the linear regime (Figure 4) and at the non-linear regime (Figure 5).

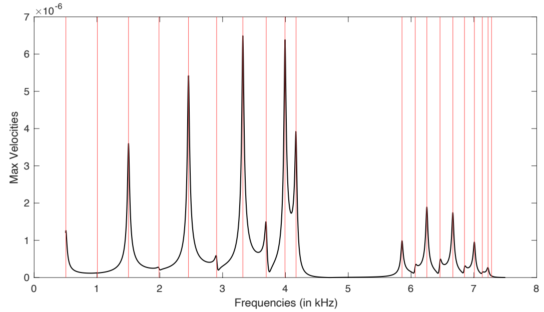

Figure 3: Frequency continuation graph with eigenfrequencies from linear problem

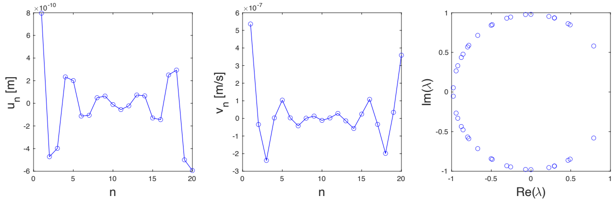

Figure 4: $f = 5\ kHz, a = 1nm$ case, the magnitudes of eigenvalues are within the real/complex unit circle

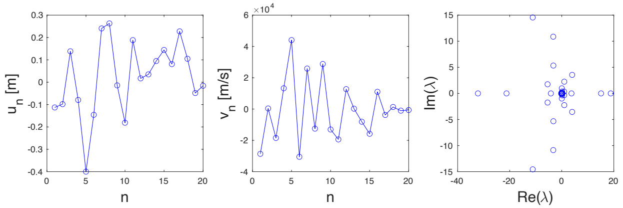

Figure 5: $f = 5\ kHz, a = 1\mu m$ case, the magnitudes of eigenvalues are away from the real/complex unit circle

Future Directions

- Perform a frequency continuation with value of the amplitude at 10^{-6} and identify the stability characteristics of time-periodic solutions.
- Embed a defect (or impurity) at the center of the chain and explore the configuration space of time-period solutions.
- The defect plays the role of a PZT sensor. Stable, time-periodic solutions in that case (with high amplitude) will be suitable for energy harvesting applications.

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